

Wednesday, Sep 10, Lecture 6: Vector spaces and bases.

Last time...

Systems of linear equations $\Leftrightarrow A\vec{x} = \vec{b}$

Solution structure

rref: No solution

rref([A:b]) (example) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Inconsistent.

1 solution $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

1 solution
(0 free variables
= 0 dim soln space)

Many solutions $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1 free variable
= 1 dimensional
soln space

Notation: empty space = fill with 0's.
in matrix

Thm:

Fix an $m \times n$ matrix A and $m \times 1$ vector $\vec{b}$.

The matrix equation $A\vec{x} = \vec{b}$ has...

- No solution iff the last column of rref([A:b]) is a pivot column:

$$\text{rref}([A:b]) = \begin{pmatrix} \text{wavy} & \vdots & \text{wavy} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

↑
pivot col

• Inconsistent system

- Exactly one solution iff every column except the last column is a pivot column:

$$\text{rref}([A:b]) = \begin{pmatrix} 1 & & & & \vdots & \text{wavy} \\ & 1 & & & & \vdots \\ & & \ddots & & & \vdots \\ & & & 1 & & \vdots \\ 0 & 0 & \dots & 0 & & 0 \\ 0 & 0 & \dots & 0 & & 0 \end{pmatrix}$$

No free variables + consistent system

- Many solutions iff the last column is not a pivot column (consistent), and some other column is not a pivot column (at least 1 free variable)

$$\text{ref}([A:b]) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}$$

↑ free variable
 ↑ consistent

⌈ MAT 54 reminder:

Def: Let $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ represent a linear map
 (soon: once we pick a basis, every linear map corresponds to a matrix)

- The null space (or kernel) of A is

$$\text{null}(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \} \subseteq \mathbb{R}^n$$

- Tells us about uniqueness of solutions.
- # free variables in $\text{ref}[A:\vec{b}] \Leftrightarrow \text{dimension of } \text{null}(A)$.

Unique: 0 free variables $\Leftrightarrow \dim(\text{null}(A)) = 0$

Many Soln: $1 \leq \# \text{ free variables} \Leftrightarrow \dim(\text{null}(A)) \geq 1$

- The range of A (aka image) is

$$\text{ran}(A) = \left\{ \vec{b} \in \mathbb{R}^m : \begin{array}{l} \text{there is a solution} \\ \vec{x} \text{ s.t. } A\vec{x} = \vec{b} \end{array} \right\} \subseteq \mathbb{R}^m$$

- Tells us about existence of solutions.

$A\vec{x} = \vec{b}$ is consistent (a soln exists) $\Leftrightarrow \vec{b} \in \text{ran}(A)$

$A\vec{x} = \vec{b}$ is not consistent (no soln) $\Leftrightarrow \vec{b} \notin \text{ran}(A)$

Vector Spaces

Vectors are defined by what they do:

- Add: if $v \in V$ and $w \in V$, then
 $v + w \in V$

- Scalar multiply: if $a \in \mathbb{R}$ (or $\mathbb{C}$) and $v \in V$, then
 $av \in V$.

Vector space over $\mathbb{R}$ (or $\mathbb{C}$) = $\{\text{vectors}\}$

Crucially, these operations cannot allow you to "leave" V .

Ex:

• $V = \mathbb{R}^3$ is a vector space.

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \in V$$

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_2 \\ ax_3 \end{bmatrix} \in V \quad \text{if } a \in \mathbb{R}.$$

• $V = \mathbb{R}^3 \setminus \{\vec{0}\}$ is not a vector space:

$$\vec{0} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \vec{0} \notin V$$

• $V = \{\text{polynomials of degree } \leq 2\}$ is a vector space:

$$p(x) = a_2 x^2 + a_1 x + a_0$$

$$q(x) = b_2 x^2 + b_1 x + b_0$$

$$\text{Add: } p(x) + q(x) = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \in V$$

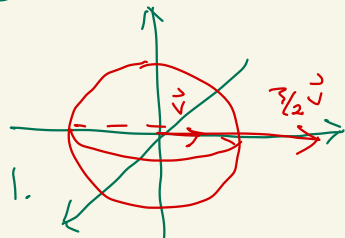
$$\text{Scalar: } r \cdot p(x) = r a_2 x^2 + r a_1 x + r a_0 \in V$$

• $V = \{\vec{v} \in \mathbb{R}^3 : \|\vec{v}\| := v_1^2 + v_2^2 + v_3^2 \leq 1\}$ $\mathbb{R}^3$

is not a vector space, since eg.

$$\vec{v} = \frac{1}{2}\hat{i} \in V, \text{ but}$$

$$3\vec{v} = \frac{3}{2}\hat{i} \text{ has norm } \|3\vec{v}\| = \frac{3}{2} > 1.$$



• Many physical quantities (eg. gravitational/electric field, quantum observables) live in vector spaces!

Span

- Linear combinations:

if $v_1, \dots, v_p \in V$ are vectors, and $c_1, \dots, c_p \in \mathbb{R}$ are scalars, then

$$y = c_1 v_1 + c_2 v_2 + \dots + c_p v_p \in V$$

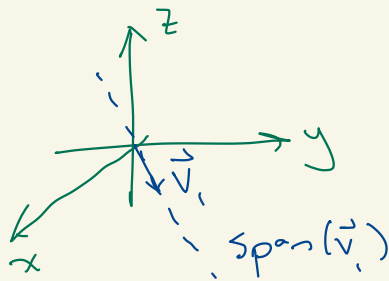
is called a linear combination.

Def: Span

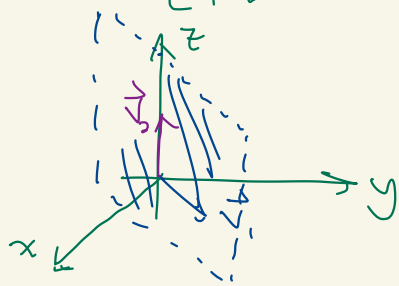
$$\begin{aligned} \text{Span}(v_1, \dots, v_p) &= \{ y = c_1 v_1 + \dots + c_p v_p \mid c_i \in \mathbb{R}, v_i \in V \} \\ &= \{ \text{linear combos of } v_1, \dots, v_p \}. \end{aligned}$$

Ex:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3, \quad \text{span}(\vec{v}_1) = \{ a \cdot \vec{v}_1 : a \in \mathbb{R} \}$$



$$\begin{aligned} \vec{v}_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3, \quad \text{span}(\vec{v}_1, \vec{v}_2) = \{ a \vec{v}_1 + b \vec{v}_2 : a, b \in \mathbb{R} \} \\ &= \left\{ \begin{bmatrix} a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}. \end{aligned}$$



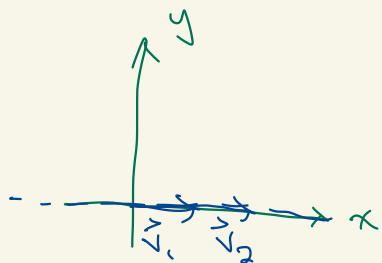
• Geometrically, clear that
dimension $\leq$ # spanning vectors.

• Hope: dimension = # spanning vectors.

But this doesn't work:

Ex:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^2, \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \mathbb{R}^2$$



$$\begin{aligned} \text{span}(\vec{v}_1, \vec{v}_2) &= \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\} \\ &= \text{span}(\vec{v}_1). \end{aligned}$$

What went wrong? $\vec{v}_2$ was redundant:

$$\vec{v}_2 = 2\vec{v}_1, \text{ so } \vec{v}_2 \in \text{span}(\vec{v}_1).$$

This is the notion of linear dependence.

Linear independence

Def:

A set of vectors $\vec{v}_1, \dots, \vec{v}_n$ is linearly independent if the only solution to the eqn

$$a_1 \vec{v}_1 + \dots + a_n \vec{v}_n = \vec{0}$$

is $a_1 = a_2 = \dots = a_n = 0$.

Otherwise, say $\vec{v}_1, \dots, \vec{v}_n$ is linearly dependent.

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is linearly dependent,

since $a_1 \vec{v}_1 + a_2 \vec{v}_2 = \vec{0}$

has nonzero solutions: if $a_1 = 1$ and $a_2 = -\frac{1}{2}$, then

$$1 \cdot \vec{v}_1 - \frac{1}{2} \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is one example.

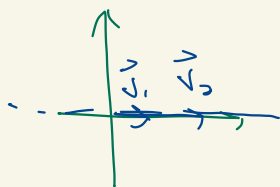
A more algorithmic approach:

divide row by a_1 if $a_1 \neq 0$.

$$\text{ref} \left(\begin{bmatrix} a_1 & a_2 & : & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & \frac{a_2}{a_1} & : & 0 \end{bmatrix}$$

1 free variable $\left(\frac{a_1}{a_2} \right) = \infty$ many solutions to $a_1 \vec{v}_1 + a_2 \vec{v}_2 = \vec{0}$.

Geometrically:



$\vec{v}_2$ is redundant!